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Integral Equations of Volterra Type

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The behavior of exact solutions to Volterra linear and non-linear integral equations with negative or positive, monotone kernels is studied. It includes properties such as the number of zeroes, boundedness and monotonicity of the solutions on the infinite interval.

INTRODUCTION

Integral equations of Volterra type arise in a wide variety of areas in Physical and Biological sciences—see for example [1, 5]. Usually they describe processes such as the "renewal" process, in which the unknown function at any time is expressible in terms of its values in the past.

In this work, properties of solutions to integral equations of the form

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\, G[f(\tau), \tau]\, d\tau$$

are studied. The functions $\phi(t)$, $K(t)$ and $G(x, t)$ are given. The kernel $K(t)$ is assumed to be monotone and satisfies certain other properties. The source term $\phi(t)$ satisfies certain properties which may depend on $K(t)$. Results on properties of $f(t)$ are obtained; some of these results are similar to those of Friedman [3], but some sufficient conditions are independent of his.

0. PRELIMINARIES

0.1. *Existence and Uniqueness theorems*

For future reference, we list below a few well-known existence and uniqueness theorems for solutions to Volterra integral equations of the form

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\, G[f(\tau), \tau]\, d\tau. \tag{0.1.1}$$

For proofs of them, see for example [1, 3, 5].

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THEOREM 0.1.1. (a) *If* (1) $\phi(t)$ *is continuous for* $0 \leqslant t < \infty$; (2) $G(x, t)$ *is continuous for* $-\infty < x < \infty$, $0 \leqslant t < \infty$; (3) $K(t)$ *is continuous for* $0 < t < \infty$ *and* $K \in L(0, 1)$, *i.e.* $\int_0^1 |K(t)| \, dt < \infty$, *then there exists a continuous solution of* (0.1.1) *for* $0 \leqslant t \leqslant T$, *for some* $T > 0$.

(b) *If* $G(x, t)$ *is, in addition, locally Lipschitz continuous in* x, *uniformly with respect to* t *in bounded sets, then the solution is unique.*

THEOREM 0.1.2. *If in addition to the assumptions made in Theorem* 0.1.1, *we assume that for any* $T > 0$ *and for any solution* $f(t)$ *of* (0.1.1) *in* $0 \leqslant t \leqslant T$, *the inequality* $|f(t)| \leqslant M$ *holds, where* M *is independent of* T *and of the particular solution* $f(t)$, *then there exists a unique solution of* (0.1.1) *in* $0 \leqslant t < \infty$.

0.2. *A Convolution Theorem*

Of special interest are equations in which the integral is a convolution. If the equation is linear

$$f(t) = \phi(t) + \int_0^t K(t - \tau) f(\tau) \, d\tau = \phi(t) + K * f \qquad (0.2.1)$$

(we will use this notation in sequel), then one can relate the solution to the solution of the equation

$$g(t) = 1 + \int_0^t K(t - \tau) g(\tau) \, d\tau.$$

THEOREM 0.2.1. *If* (1) $\phi'(t)$ *exists for* $0 \leqslant t \leqslant T$, $\int_0^T |\phi'(t)| \, dt < \infty$ *and* (2) $\int_0^T |K(t)| \, dt < \infty$, *then a solution to* (0.2.1) *is given by*

$$f(t) = g(t) \phi(0) + \int_0^t g(t - \tau) \phi'(\tau) \, d\tau, \qquad \text{for } 0 \leqslant t \leqslant T. \qquad (0.2.2)$$

(*See* [1] *for the proof.*)

Therefore if $g(t)$ is known, so is $f(t)$. Or if properties of $g(t)$ are known, we may be able to obtain certain properties of $f(t)$ through (0.2.2).

0.3. *The Neumann Series*

One method of obtaining a solution to (0.2.1) or more generally to

$$f(t) = \phi(t) + \int_0^t K(t, \tau) f(\tau) \, d\tau$$

is by iteration. One starts by setting $f_0(t) = \phi(t)$ and defines $f_n(t) = \phi(t) + \int_0^t K(t, \tau) f_{n-1}(\tau) \, d\tau$. If the iteration converges, one would hope that it would converge to the solution. The same result would be obtained if one introduces a parameter λ

$$f(t) = \phi(t) + \lambda \int_0^t K(t, \tau) f(\tau)\, d\tau$$

and expands f as a power series in λ. This series is called the Neumann Series. Its convergence is given by the following theorem.

THEOREM 0.3.1. *If* (1) $K(t, \tau)$ *is quadratically integrable in the square* $(0 \leqslant t \leqslant T, 0 \leqslant \tau \leqslant T)$ *where* $T > 0$ *is a constant and* (2) $\phi(t) \in L_2(0, T)$, *then the* L_2*-solution* $f(t)$ *is given by*

$$f(t) = \sum_{n=0}^{\infty} \lambda^n \psi_n(t)$$

where $\psi_0(t) = \phi(t)$,

$$\psi_n(t) = \int_0^t K_n(t, \tau)\, \phi(\tau)\, d\tau \qquad (n = 1, 2, 3, \ldots)$$

where $K_1(t, \tau) = K(t, \tau)$,

$$K_{n+1}(t, \tau) = \int_\tau^t K(t, s)\, K_n(s, \tau)\, ds \qquad (n = 1, 2, 3, \ldots).$$

(*See* [5] *for the proof.*)

Therefore if $\lambda > 0$, K and ϕ are positive functions, then we can conclude from the Neumann Series that the solution $f(t)$ is positive on $[0, T]$.

1. Properties of Solutions

1.1. *Equivalence Theorems*

The idea behind the following equivalence theorem is to reduce the original kernel to an integrable kernel or to a kernel which is better behaved at ∞, by possibly transferring certain bad behavior of the kernel to the source teım.

THEOREM 1.1.1. *If* (1) $K(t) \in C^1$ *for* $0 \leqslant t < \infty$ *and* (2) $\phi(t)$ *is locally integrable, then the following two integral equations are equivalent:*

$$f(t) = \phi(t) - \int_0^t K(t - \tau) f(\tau)\, d\tau, \tag{1.1.1}$$

$$f(t) = \phi(t) - \int_0^t L(t - \tau) f(\tau)\, d\tau, \tag{1.1.2}$$

where

$$L(t) = g'(t) + ag(t) + \int_0^t g(t-\tau)\, K'(\tau)\, d\tau,$$

$$\psi(t) = \phi(t) + \int_0^t g'(t-\tau)\, \phi(\tau)\, d\tau,$$

where $a = K(0)$, g *is any function such that*

$$g(t) \in C^1[0, \infty) \qquad \textit{and} \qquad g(0) = 1.$$

Proof. Suppose $f(t)$ satisfies (1.1.2). Taking $g(t) = 1$, we have

$$\begin{aligned} L(t) &= a + K(t) - a = K(t), \\ \psi(t) &= \phi(t), \end{aligned}$$

thus $f(t)$ satisfies (1.1.1).

Conversely, suppose $f(t)$ satisfies (1.1.1). Note first that

$$\begin{aligned} L(t) &= g'(t) + ag(t) + g(t-\tau)\, K(\tau) \Big|_0^t + \int_0^t g'(t-\tau)\, K(\tau)\, d\tau \\ &= g'(t) + K(t) + \int_0^t g'(t-\tau)\, K(\tau)\, d\tau, \end{aligned}$$

so

$$\begin{aligned} \int_0^t L(t-\tau) f(\tau)\, d\tau &= \int_0^t g'(t-\tau) f(\tau)\, d\tau + \int_0^t K(t-\tau) f(\tau)\, d\tau \\ &\quad + \int_0^t \left[\int_0^{t-\tau} g'(t-\tau-s)\, K(s)\, ds\right] f(\tau)\, d\tau \\ &= \int_0^t g'(t-\tau) f(\tau)\, d\tau + \phi(t) - f(t) \\ &\quad + \int_0^t \left[\int_0^{t-\tau} g'(t-\tau-s)\, K(s)\, ds\right] f(\tau)\, d\tau \end{aligned}$$

where

$$\begin{aligned} &\int_0^t \left[\int_0^{t-\tau} g'(t-\tau-s)\, K(s)\, ds\right] f(\tau)\, d\tau \\ &\qquad = \int_0^t \left[\int_\tau^t g'(t-r)\, K(r-\tau)\, dr\right] f(\tau)\, d\tau, \qquad \tau + s = r, \\ &\qquad = \int_0^t \left[\int_0^r K(r-\tau) f(\tau)\, d\tau\right] g'(t-r)\, dr \\ &\qquad = \int_0^t (\phi(r) - f(r))\, g'(t-r)\, dr; \end{aligned}$$

therefore

$$\int_0^t L(t-\tau) f(\tau)\, d\tau = \phi(t) + \int_0^t g'(t-\tau)\,\phi(\tau)\, d\tau - f(t).$$

Remarks. (1) *Extension to Banach spaces.* This theorem immediately extends to the case where the functions $\phi(t)$ and $f(t)$ in (1.1.1) take values in a Banach space. The proof is along the same line to the scalar case; the interchange of order of integration is justified under the assumptions that (K) $K(0) > 0$; $K \in C^1[0, T)$; $K'(t)$ is absolutely continuous in $[T_0, T)$; $K'' \in L^p(0, T)$ for some $p > 1$ and (ϕ) $\phi'(t)$ exists for all $0 \leqslant t < T$; ϕ' is uniformly Holder continuous. (See [4]).

(2) *Extension to non-linear integral equations.* Theorem 1.1.1 extends to the case for the non-linear integral equation

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\, G[f(\tau), \tau]\, d\tau. \tag{1.1.3}$$

In this case, the new kernel $L(t)$ is as in Theorem 1.1.1 and the new source term $\psi(t)$ is

$$\psi(t) = \phi(t) + \int_0^t g'(t-\tau)\,\phi(\tau)\, d\tau + \int_0^t g'(t-\tau)\,[f(\tau) - G(f(\tau), \tau)]\, d\tau.$$

When $G(x, y) = xh(y)$, then (1.1.3) becomes

$$f(t) = \phi(t) - \int_0^t K(t-\tau) f(\tau)\, h(\tau)\, d\tau$$

for which, as we know, the solution cannot be obtained by the use of Laplace transform

(3) *Examples.* In Theorem 1.1.1, $g(t)$ can be, for examples, $\exp(-\gamma t)$ where γ is any constant; or $\exp(-\int_0^t K(\tau)\, d\tau)$. The former choice would be used in Section 1.2.

LEMMA 1.1.1. *Let $g(t) = \exp(-\int_0^t K(\tau)\, d\tau)$ in Theorem 1.1.1. If $K(t) > 0$ and $\int_0^\infty K(t)\, dt < \infty$, then $|\int_0^\infty L(t)\, dt| < 1$.*

Proof. Let $\int_0^\infty K(t)\, dt = A$, then $g(\infty) = e^{-A}$. By Theorem 1.1.1,

$$L(t) = g'(t) + ag(t) + \int_0^t g(t-\tau)\, K'(\tau)\, d\tau,$$

so $L(t) = g'(t) + K(t) + \int_0^t g'(t-\tau)\, K(\tau)\, d\tau$. Therefore

$$\int_0^T L(t)\, dt = g(T) - 1 + \int_0^T K(t)\, dt + \int_0^T \left[\int_0^t g'(t-\tau)\, K(\tau)\, d\tau\right] dt$$

where

$$\int_0^T \left[\int_0^t g'(t-\tau)\,K(\tau)\,d\tau\right] dt = \int_0^T \left[\int_\tau^T g'(t-\tau)\,dt\right] K(\tau)\,d\tau$$
$$= \int_0^T (g(T-\tau)-1)\,K(\tau)\,d\tau,$$

and so

$$\int_0^\infty L(t)\,dt = g(T) - 1 + \int_0^T g(T-\tau)\,K(\tau)\,d\tau,$$

$$\int_0^\infty L(t)\,dt = e^{-A} - 1 + e^{-A}\cdot A \qquad \text{(note that } g \text{ is monotonic and bounded)}$$
$$= (1+A)\,e^{-A} - 1$$

where $0 < (1+A)\,e^{-A} < 1$, so $|\int_0^\infty L(t)\,dt| < 1$.

Given below is an immediate use of Theorem 1.1.1 with Remark (3) to show the positivity of the solution to the integral equation

$$f(t) = 1 - \int_0^t K(t-\tau) f(\tau)\,d\tau$$

where the kernel may be infinite at the origin. (See [5].)

THEOREM 1.1.2. *If* (1) $K(t) > 0$ *on* $(0, \infty)$, $K(0)$ *needs not be finite and* (2) $K'(t) < 0$ *on* $(0, \infty)$, *then* $f(t) > 0$ *for* $t > 0$.

Proof. The equation is $f(t) = 1 - K * f$.

By Theorem 1.1.1, $f(t) = g(t) - L * f$ where

$$L(t) = g'(t) + K(t) + g' * K.$$

Note that with $L(t)$ in the above form, $K(0)$ need not be finite.

Now choose $g(t) = \exp(-\int_0^t K(\tau)\,d\tau)$, then $g'(t) = -g(t)\,K(t) < 0$ and $g'(t) + K(t) = (1 - g(t))\,K(t) < 0$, so $L(t) < 0$ and $f(t) > 0$ by the Neumann Series.

EXAMPLE. $K(t) = M(t)/t^\alpha$ where $M(t)$ is continuous for all t, $M(t) > 0$, $M'(t) < 0$ and $0 \leqslant \alpha < \frac{1}{2}$.

1.2. *Linear Integral Equations*

With the use of the Equivalence Theorem 1.1.1, certain properties of integral equation of the form

$$f(t) = 1 - \int_0^t K(t-\tau) f(\tau)\,d\tau$$

can be derived. It would be assumed that all involved derivatives of $K(t)$ are continuous for $0 \leqslant t < \infty$.

THEOREM 1.2.1. *If* (1) $K(0) = a < 0$; (2) $K'(0) = b$; (3) $K''(t) < 0$ *and* (4) $a^2 \geqslant 4b$, *then* $f^{(n)}(t) > 0$, *for* $n = 0, 1, 2, 3$.

Proof. The case for $b \leqslant 0$ is trivial, so assume $b > 0$. The equation is $f(t) = 1 - K * f$.

By Theorem 1.1.1, $f(t) = e^{-\gamma t} - L * f$, where

$$L(t) = (a - \gamma)\, e^{-\gamma t} + K' * e^{-\gamma t},$$

so $L'(t) = (\gamma^2 - a\gamma + b)\, e^{-\gamma t} + K'' * e^{-\gamma t}$.

Now choose $\gamma = \frac{1}{2}(a + (a^2 - 4b)^{1/2})$, then $L'(t) < 0$. But $L(0) = a - \gamma < 0$, so $L(t) < 0$. It follows from the Naumann series that $f(t) > 0$. Further,

$$\begin{aligned} f'(t) &= -\gamma e^{-\gamma t} - L(t) - L * f', \\ f''(t) &= \gamma^2 e^{-\gamma t} - L'(t) + aL(t) - L * f'', \\ f'''(t) &= -\gamma^3 e^{-\gamma t} - L''(t) + aL'(t) - (a^2 - b)\, L(t) - L * f''', \end{aligned}$$

where $L''(t) = K''(t) - \gamma K'' * e^{-\gamma t}$, which is negative, therefore $f'(t)$, $f''(t)$ and $f'''(t)$ are also positive.

EXAMPLE. $K(t) = a + bt$, $a < 0$, $b > 0$ and $a^2 \geqslant 4b$.

Using Theorem 1.2.1, the following theorem for positive decreasing kernels can be proved.

THEOREM 1.2.2. *If* (1) $K(t) > 0$; (2) $K'(t) < 0$; (3) $K''(t) > 0$;

(4) $K'''(t) < 0$ *and* (5) $c \leqslant \gamma_0(a - \gamma_0)^2/4$, *where* $\gamma_0 = (a + 2(a^2 - 3b)^{1/2})/3$, $a = K(0)$, $b = K'(0)$, $c = K''(0)$, *then* (1) $0 < f(t) \leqslant 1$; (2) $f'(t)$ *has at most one zero and* (3) $f''(t) > 0$.

Proof. The equation is $f(t) = 1 - K * f$. By Theorem 1.1.1, $f(t) = e^{-\gamma t} - L * f$. Consider the equation $g(t) = 1 - L * g$. By Theorem 0.2.1, we can express $f(t)$ in terms of $g(t)$

$$f(t) = g(t) - \gamma e^{-\gamma t} * g,$$

so

$$f'(t) = g'(t) - \gamma g(t) + \gamma^2 e^{-\gamma t} * g,$$

and $\gamma f + f' = g'$.

Now to obtain the properties of $g(t)$, we make use of Theorem 1.2.1. The equation is

$$g(t) = 1 - L * g$$

where

$$L(t) = (a - \gamma) e^{-\gamma t} - K' * e^{-\gamma t},$$

so $L'(t) = (\gamma^2 - a\gamma + b) e^{-\gamma t} + K'' * e^{-\gamma t}$ and $L''(t) = -\gamma(\gamma^2 - a\gamma + b) e^{-\gamma t} + ce^{-\gamma t} + K''' * e^{-\gamma t}$. Now let $\gamma = (a + 2(a^2 - 3b)^{1/2})/3$, which satisfies

$$(a - \gamma)^2 = 4(\gamma^2 - a\gamma + b),$$

so $L^2(0) = 4L'(0)$ and noting that $a < \gamma$, we see that $L(0) < 0$ and $L'(0) > 0$. Finally, since $K'''(t) < 0$ and $-\gamma(a - \gamma)^2/4 + c \leqslant 0$, we have $L''(t) < 0$. By Theorem 1.2.1, $g^{(n)}(t) > 0$, $n = 0, 1, 2, 3$.

Now we can obtain the properties of $f(t)$. Since $L(t) < 0$, we have $f(t) > 0$ and so $0 < f(t) \leqslant 1$.

From

$$\gamma f + f' = g',$$

we get

$$\gamma f' + f'' = g'' > 0$$

and since $f'(0) = -a < 0$, $f'(t)$ has at most one zero. Further, $\gamma f'' + f''' = g''' > 0$ and since $f''(0) = a^2 - b > 0$, $f''(t) > 0$.

EXAMPLE. $K(t) = 1/(10 + t)$.

Remark. To show that $0 < f(t) \leqslant 1$, only the assumptions $K(t) > 0$ and $K'(t)$ are needed.

Theorem 1.2.2 can be applied to derive the following result for positive increasing kernels.

THEOREM 1.2.3. *If* (1) $K(t) > 0$; (2) $K'(t) > 0$; (3) $K''(t) < 0$; (4) $K'''(t) > 0$; (5) $K^{(4)}(t) < 0$ *and* (6) $a^3 \geqslant 4ab - 8c + 16(d/a)$ *and* $a^3 \leqslant 4ab - 8c + 2\gamma_0'(a/2 - \gamma_0')^2$, *where* $\gamma_0' = a/b + \frac{2}{3}(a^2 - 3b)^{1/2}$, $a = K(0)$, $b = K'(0)$, $c = K''(0)$, $d = K'''(0)$, *then* (1) $|f(t)| \leqslant 1$, (2) $f(t)$ *has at most* 2 *zeros and* (3) $f'(t)$ *has at most one zero.*

Proof. The equation is $f(t) = 1 - K * f$. By Theorem 1.1.1, $f(t) = e^{-\gamma t} - L * f$. With $\gamma = \gamma_0 = a/2$,

$$L(t) = \frac{a}{2} e^{-\gamma_0 t} + K' * e^{-\gamma_0 t} > 0,$$

$$L'(t) = \left(-\frac{a^2}{4} + b\right) e^{-\gamma_0 t} + K'' * e^{-\gamma_0 t} < 0, \qquad \text{since} \qquad a^2 > 4b.$$

Consider the equation, $g(t) = 1 - L * g$, we have $0 < g(t) \leqslant 1$ by Theorem 1.2.2.

By Theorem 0.2.1, $f(t) = g(t) - \gamma_0 e^{-\gamma_0 t} * g$, so $|f(t)| \leqslant 1$.

We now show that $L(t)$ also satisfies the other assumptions of Theorem 1.2.2.

$$L''(t) = \left(\frac{a^3}{8} - \frac{ab}{2} + c\right) e^{-\gamma_0 t} + K''' * e^{-\gamma_0 t}.$$

Since $a^3 > 4ab - 8c$, $L''(t) > 0$.

$$L'''(t) = \left(-\frac{a^4}{16} + \frac{a^2 b}{4} - \frac{aC}{2} + d\right) e^{-\gamma_0 t} + K^{(4)} * e^{-\gamma_0 t}.$$

By the first of the two inequalities assumed, $L'''(t) < 0$.

Theorem 1.2.2 also requires that the following inequality be satisfied:

$$L''(0) \leqslant \frac{\gamma_0' \left(\frac{a}{2} - \gamma_0'\right)^2}{4}, \qquad \gamma_0' = \frac{\frac{a}{2} + 2\left(\left(\frac{a}{2}\right)^2 - 3\left(-\frac{a^2}{4} + b\right)\right)^{1/2}}{3}.$$

But $L''(0) = a^3/8 - ab/2 + c$, so we need that

$$a^3 \leqslant 4ab - 8c + 2\gamma'_0 \left(\frac{a}{2} - \gamma'_0\right)^2, \qquad \gamma'_0 = \frac{a}{6} + \frac{2}{3}(a^2 - 3b)^{1/2}.$$

We can now conclude further that $g'(t)$ has at most one zero and $g''(t) > 0$.

To obtain the other properties of $f(t)$, we note as before that

$$\gamma_0 f + f' = g',$$
$$\gamma_0 f' + f'' = g''$$

and $f''(0) = -a < 0$, so $f'(t)$ has at most one zero and $f(t)$ has at most 2 zeroes.

Remark. The assumptions $K(t) > 0$, $K'(t) > 0$, $K''(t) < 0$ and $a^2 \geqslant 4b$ are sufficient for showing that $|f(t)| \leqslant 1$.

The next theorem is also for positive increasing kernels. Use is made of Theorem 2 in [3] besides Theorem 1.1.1.

THEOREM 1.2.4. *If* (1) $K(t) > 0$; (2) $K'(t) > 0$; (3) $K''(t) < 0$; (4) K''/K' *is non-decreasing and* (5) $a + c/b \geqslant 2b^{1/2}$, $a = K(0)$, $b = K'(0)$, $c = K''(0)$, *then* (1) $|f(t)| \leqslant 1$ *and* (2) $f(t)$ *has at most one zero.*

Proof. The equation is $f(t) = 1 - K * f$. By Theorem 1.1.1, $f(t) = e^{-\gamma t} - L * f$.

Consider the equation $g(t) = 1 - L * g$, in order to apply Theorem 2 in [3], we have to show that $L(t)$ satisfies three assumptions, namely, $L(t) > 0, L'(t) < 0$ and $\log L(t)$ is convex. We have

$$\begin{aligned} L(t) &= (a-\gamma)\, e^{-\gamma t} + K' * e^{-\gamma t}, \\ L'(t) &= K'(t) - \gamma L(t), \\ L''(t) &= K''(t) - \gamma K'(t) + \gamma^2 L(t); \end{aligned}$$

therefore

$$LL'' - L'^2 = LK'' - K'^2 + \gamma LK'.$$

Let $N(t) = L(t)L''(t) - L'^2(t)$; we want to show $N(t) \geqslant 0$. With $L(t)$ substituted in, we have

$$\begin{aligned} e^{\gamma t}N(t) = {} & \gamma(a-\gamma)\, K' + \gamma K' \int_0^t e^{\gamma\tau} K(\tau)\, d\tau \\ & + (a-\gamma)\, K'' + K'' \int_0^t e^{\gamma\tau} K'(\tau)\, d\tau - e^{\gamma t} K'^2 \end{aligned}$$

where

$$\int_0^t e^{\gamma\tau} K'(\tau)\, d\tau = \frac{1}{\gamma} e^{\gamma t} K'(t) - \frac{1}{\gamma} b - \frac{1}{\gamma} \int_0^t e^{\gamma\tau} K''(\tau)\, d\tau;$$

therefore

$$\begin{aligned} e^{\gamma t}N(t) = {} & \gamma(a-\gamma)\, K' + (a-\gamma)\, K'' \\ & - bK' + \int_0^t e^{\gamma\tau}[K''(t)\, K(\tau) - K'(t)\, K''(\tau)]\, d\tau, \end{aligned}$$

$$\frac{e^{\gamma t}N(t)}{K'(t)} = \gamma(a-\gamma) + (a-\gamma)\frac{K''}{K'} - b + \int_0^t e^{\gamma\tau} \frac{K''(t)\, K'(\tau) - K'(t)\, K''(\tau)}{K'(t)}\, d\tau,$$

where

$$\begin{aligned} & \gamma(a-\gamma) + (a-\gamma)\frac{K''}{K'} - b \\ & \quad \geqslant \gamma(a-\gamma) + (a-\gamma)\frac{c}{b} - b, \qquad \text{if} \quad \gamma < a \\ & \quad = \gamma(a-\gamma) + \gamma^2 - a\gamma + b - b, \qquad \text{if} \quad (a-\gamma)\frac{c}{b} = \gamma^2 - a\gamma + b \\ & \quad = 0. \end{aligned}$$

To solve for γ,

$$\gamma^2 + \left(-a + \frac{c}{b}\right)\gamma + \left(b - \frac{ac}{b}\right) = 0$$

$$\gamma = \frac{1}{2}\left(a - \frac{c}{b} - \left(\left(a + \frac{c}{b}\right)^2 - 4b\right)^{1/2}\right)$$

so requiring that $|a + c/b| \geqslant 2b^{1/2}$. Clearly $\gamma > 0$. We also want $\gamma < a$, i.e.

$$a - \frac{c}{b} - \left(\left(a + \frac{c}{b}\right)^2 - 4b\right)^{1/2} < 2a$$

$$-\left(\left(a + \frac{c}{b}\right)^2 - 4b\right)^{1/2} < a + \frac{c}{b}$$

which requires that $a + c/b > 0$. Therefore $a + c/b \geqslant 2b^{1/2}$ is assumed. Further,

$$\frac{K''(t)}{K'(t)}(K'(\tau) - K''(\tau)) \geqslant \frac{K''(\tau)}{K'(\tau)}(K'(\tau) - K''(\tau)) = 0,$$

therefore $e^{\gamma t}N(t)/K'(t) \geqslant 0$. Now $N(t) \geqslant 0$ implies that $\log L(t)$ is convex. Since $a - \gamma > 0$, $L(t) > 0$. Since $\gamma^2 - a\gamma + b < 0$, $L'(t) < 0$.

By Theorem 2 in [3], we conclude that $g'(t) < 0$. By Theorem 1.2.2, we have $0 < g(t) \leqslant 1$.

To obtain the properties of $f(t)$, we apply as before the Convolution Theorem and write

$$f(t) = g(t) - \gamma e^{-\gamma t} * g$$

so $|f(t)| < 1$. Again, we have

$$\gamma f + f' = g' < 0$$

so $f(t)$ has at most one zero.

EXAMPLE. $K(t) = \ln(e^2 + t)$.

So far Theorem 1.1.1 has been used with $g(t) = \exp(-\gamma t)$. The next theorem for positive decreasing kernels can be derived using a different choice for $g(t)$.

THEOREM 1.2.5. *If* (1) $K(t) > 0$; (2) $K'(t) < 0$ *and* (3) $-K'(t) > a^2 \exp(-\int_0^t K(\tau)\,d\tau)$, *where* $a = k(0)$, *then* (1) $0 < f(t) \leqslant 1$; (2) $f'(t)$ *has at most one zero and* (3) $|f'(t)| < 2a$.

Proof. The equation is $f(t) = 1 - K * f$. By Theorem 1.1.1 with $g(t) = \exp(-\int_0^t K(\tau)\,d\tau)$, we have

$$f(t) = g(t) - L * f$$

where

$$\begin{aligned} L(t) &= g'(t) + ag(t) + K' * g \\ &< g'(t) + ag(t) + g(t)\,(K(t) - a) \\ &= 0, \qquad \text{since} \qquad g'(t) = -K(t)\,g(t), \end{aligned}$$

so $f(t) > 0$ by the Neumann Series. From $f(t) = 1 - K * f$, therefore $0 < f(t) \leqslant 1$.

To show that $f'(t)$ has at most one zero, note that

$$\begin{aligned} f'(t) &= g'(t) - L(t) - L * f' \\ &= -ag(t) - K' * g - L * f', \\ f''(t) &= -ag'(t) - K'(t) - K' * g' + aL(t) - L * f'' \\ &= -K'(t) - K' * g' + a^2 g(t) + aK' * g - L * f''; \end{aligned}$$

therefore

$$\begin{aligned} 2af'(t) + f''(t) &= -a^2 g(t) - aK' * g - K'(t) - K' * g' - L * (2af' + f'') \\ &= -a^2 g(t) - K'(t) + K' * (K - a) g - L * (2af' + f'') \end{aligned}$$

where the source term is positive and so $2af'(t) + f''(t) > 0$. Since $f'(0) = -a$, $f'(t)$ has at most one zero.

To show that $f'(t)$ is bounded, from

$$\begin{aligned} f(t) &= 1 - K * f, \\ f'(t) &= -af(t) - K' * f, \end{aligned}$$

so

$$\begin{aligned} |f'(t)| &\leqslant a - (K(t) - a) \\ &\leqslant 2a. \end{aligned}$$

The following theorem is on positive decreasing kernels bounded away from zero.

THEOREM 1.2.6. *If* (1) $K(t) > 0$; (2) $K'(t) < 0$ *and* (3) $-K'(t) < am \exp(-\int_0^t K(\tau)\,d\tau)$, *where* $a = k(0)$, $m = K(\infty)$, *then* (1) $0 < f(t) \leqslant 1$; (2) $f'(t) < 0$ *and* (3) $|f'(t)| \leqslant 2a$.

Proof. The equation is $f(t) = 1 - K * f$. As in Theorem 1.2.5, using Theorem 1.1.1 with $g(t) = \exp(-\int_0^t K(\tau)\,d\tau)$, we see that the kernel $L(t)$ in the equivalent equation

$$f(t) = g(t) - L * f$$

is negative and so $f(t) > 0$ by the Neumann Series. From $f(t) = 1 - K * f$, therefore $0 < f(t) \leqslant 1$.

To show that $f'(t) < 0$, note as before that

$$\begin{aligned} f'(t) &= -ag(t) - K' * g - L * f', \\ f''(t) &= -K'(t) - K' * g' + a^2 g(t) + aK' * g - L * f''; \end{aligned}$$

theiefore

$$(a + m) f'(t) + f''(t)$$
$$= -amg(t) - K'(t) + K' * (K - m) g - L * [(a + m) f' + f'']$$

where the source term is negative and so $(a + m) f'(t) + f''(t) < 0$. Since $f'(0) = -a$, $f'(t) < 0$.

It can be shown as in Theorem 1.2.5 that $f'(t)$ is bounded, namely $|f'(t)| \leqslant 2a$.

Consider now the equation with a more general source term $\phi(t)$, namely

$$f(t) = \phi(t) - K * f.$$

THEOREM 1.2.7. *If* (1) *$K(t)$ satisfies the assumptions in Theorem* 1.2.6 *and* (2) *$\phi(t)$ is bounded, then* (1) *$f(t)$ is bounded;* (2) *$f(t) \leqslant \phi(t)$ and* (3) *$f \in L_1$.*

Proof. The equation is $f(t) = \phi(t) - K * f$. Let $h(t) = 1 - K * h$. By Theorem 0.2.1,

$$f(t) = \phi(t) + h' * \phi,$$

since $\phi(t)$ is bounded, $h'(t) < 0$ and $h(t)$ is bounded (by Theorem 1.2.6), we conclude that $f(t)$ is bounded.

To show the other two properties of $f(t)$, let $F(t) = \int_0^t |f(s)|\, ds$.

$$\begin{aligned} K * f &= \int_0^t K(t - \tau) f(\tau)\, d\tau \\ &= K(t - \tau) F(\tau) \Big|_0^t + \int_0^t K'(t - \tau) F(\tau)\, d\tau \\ &= aF(t) + \int_0^t K'(t - \tau) F(\tau)\, d\tau; \end{aligned}$$

therefore

$$\begin{aligned} f(t) &= \phi(t) - aF(t) - \int_0^t K'(t - \tau) F(\tau)\, d\tau \\ &\leqslant \phi(t) - aF(t) + F(t)(a - K(t)) \\ &= \phi(t) - F(t) K(t), \end{aligned}$$

so

$$F(t) < \frac{\phi(t) - f(t)}{K(t)}.$$

But $0 < F(t)$ for $t > 0$; therefore $f(t) < \phi(t)$, $t > 0$. And

$$F(t) \leqslant \frac{\phi(t) - f(t)}{K(t)} = \frac{|\phi(t) - f(t)|}{K(t)} \leqslant \frac{|\phi(t)| + |f(t)|}{K(\infty)};$$

since $F(t)$ is bounded and increasing, $F(\infty)$ exists and so $f \in L_1$.

1.3. *Non-linear Integral Equations*

In this section, some results for the non-linear integral equation

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\, G(f(\tau), \tau)\, d\tau \tag{1.3.1}$$

are obtained. It would be assumed that (1) $K \in C^1$ for $0 < t < \infty$; (2) $G(x, y)$ is continuous for $-\infty < x < \infty$ and $0 \leqslant y < \infty$ and (3) $\phi \in C^1$ for $0 \leqslant t < \infty$.

THEOREM 1.3.1. *If* (1) $K(t) > 0$, $K'(t) < 0$; (2) $G(x, y) \geqslant 0$ for $x > 0$, $G(0, y) = 0$ *and* (3) $\phi(0) > 0$, $\phi'(t) \geqslant 0$, *then* $0 < f(t) \leqslant \phi(t)$.

Proof. Suppose $f(t)$ has at least one zero and let z be the first zero. Differentiating (1.3.1), we obtain

$$f'(t) = \phi'(t) - K(0)\, G[f(t), t] - \int_0^t K'(t-\tau)\, G[f(\tau), \tau]\, d\tau,$$

so $f'(z) = \phi'(z) - \int_0^z K'(z-\tau)\, G[f(\tau), \tau]\, d\tau$. Since $f(\tau)$ is positive on $[0, z)$, the above integral is negative, so $f'(z) > 0$, which is a contradiction. Thus $f(t) > 0$.

From (1.3.1), we see now that $\phi(t) - f(t) > 0$ for $t > 0$. Therefore $0 < f(t) \leqslant \phi(t)$ for all t.

EXAMPLES. (1) $G(x, y) = x$, the linear case; (2) $G(x, y) = x^3y^2$.

Remarks. (1) If the inequalities in the assumptions (2) and (3) of Theorem 1.3.1 are replaced by $G(x, y) \leqslant 0$ for $x < 0$, $\phi(0) < 0$, $\phi'(t) \leqslant 0$, then $\phi(t) \leqslant f(t) < 0$. The proof is similar to that for Theorem 1.3.1.

(2) Theorem 1.3.1 is not true for increasing kernels; if $K(t) = 1 - e^{-t}$, $G(x, y) = x$ and $\phi(t) = 1$, then $f(t)$ is oscillatory in nature.

The next theorem is on monotonicity of the solution. It would be assumed that (1) $K \in C^1$ for $0 \leqslant t < \infty$; (2) $G_x(x, y)$ is continuous for $-\infty < x < \infty$ and (3) $\phi \in C^2$ for $0 \leqslant t < \infty$.

THEOREM 1.3.2. *If* (1) $K(t) > 0$, $K'(t) < 0$; (2) $G(x, y) = G(x)$, $G'(x) \geqslant 0$ *for* $-\infty < x < \infty$ *and* (3) $\phi'(0) < G[\phi(0)]\, a$ *where* $a = K(0)$, $\phi''(t) \leqslant G[\phi(0)] \times K'(t)$, *then* $f'(t) < 0$.

If the additional assumptions in Theorem 1.3.1 *also hold, then* $f(t)$ *is both positive and decreasing and so* $0 < f(t) \leqslant \phi(0)$.

Proof. The equation is

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\, G[f(\tau)]\, d\tau,$$

so

$$f'(t) = \phi'(t) - G[\phi(0)]\,K(t) - \int_0^t K(t-\tau)\,G'[f(\tau)]\,f'(\tau)\,d\tau$$

where

$$\phi'(0) - G[\phi(0)]\,a < 0,$$
$$\phi''(t) - G[\phi(0)]\,K'(t) \leqslant 0,$$

and $G'[f(t)]\,f'(t) \leqslant 0$ for $f'(t) < 0$; therefore by Remark (1) of Theorem 1.3.1, we have $f'(t) < 0$.

Since $f(0) = \phi(0)$, we have $0 < f(t) \leqslant \phi(0)$.

Example. $G(x) = x^3$, $\phi(t) = \ln(e + t)$, $K(t) = 1/(e + t) + C$, $C > 0$, this example satisfies the assumptions of both Theorem 1.3.1 and Theorem 1.3.2.

Corollary 1.3.2. *If in Theorem* 1.3.2, *we have* $G(x) = x$, *the linear case, then* $f(t) \leqslant \phi(t)/(1 + \int_0^t K(\tau)\,d\tau)$.

Proof. The equation is

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\,f(\tau)\,d\tau;$$

since $f(t)$ is positive and decreasing, we have

$$f(t) \leqslant \phi(t) - f(t)\int_0^t K(\tau)\,d\tau,$$

so $f(t) \leqslant \phi(t)/(1 + \int_0^t K(\tau)\,d\tau)$.

Examples. (1) $\phi(t) = \ln(e + t)$, $K(t) = 1/(e + t) + C$, $C > 0$.

(2) $\phi(t) = 2 - e^{-t}$, $K(t) = e^{-t} + C$, $C > 0$.

(3) $\phi(t) = 2\ln(1 + t) + 1/(1 + t)$, $K(t) = 2/(1 + t) - 1/(1 + t)^2 + C$, $C > 0$.

Example (3) satisfies all the assumptions except the convexity condition $KK'' - K'^2 \geqslant 0$ of Theorem 2 in [3] which therefore cannot be applied.

These results and Theorem 1.1.1 can be applied to derive certain properties of solutions to linear integral equations of the form

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\,f(\tau)\,d\tau$$

where the kernel is positive increasing. It would be assumed that (1) $K \in C^2$ for $0 \leqslant t < \infty$ and (2) $\phi \in C^2$ for $0 \leqslant t < \infty$.

THEOREM 1.3.3. *If* (1) $K(t) > 0$, $K'(t) > 0$, $K''(t) < 0$; (2) $a^2 \geqslant 4b$ *where* $a = K(0)$, $b = K'(0)$; (3) $\phi(0) > 0$, $\phi''(t) \geqslant 0$ *and* (4) *there exists a* γ *lying between the two positive numbers* $(a \pm (a^2 - 4b)^{1/2})/2$ *such that* $\phi'(0) \geqslant \gamma\phi(0)$, *then* $0 < f(t) \leqslant \phi(t)$.

Proof. The equation is $f(t) = \phi(t) - K * f$. By Theorem 1.1.1, $f(t) = \psi(t) - L * f$ where $\psi(t) = \phi(t) - \gamma e^{-\gamma t} * \phi$,

$$L(t) = (a - \gamma)\, e^{-\gamma t} + K' * e^{-\gamma t},$$

so

$$L'(t) = (\gamma^2 - a\gamma + b)\, e^{-\gamma t} + K'' * e^{-\gamma t}.$$

If γ lies between the two zeroes of $\gamma^2 - a\gamma + b$, then $L(t) > 0$ and $L'(t) < 0$. Now $\psi(0) = \phi(0) > 0$ and $\psi'(t) = \phi'(t) - \gamma e^{-\gamma t}\phi(0) - \gamma e^{-\gamma t} * \phi'$ where

$$e^{-\gamma t} * \phi' = \frac{e^{-\gamma t}}{-\gamma}\phi'(0) + \frac{\phi'(t)}{\gamma} - \frac{1}{\gamma}\, e^{-\gamma t} * \phi'';$$

therefore

$$\psi'(t) = [\phi'(0) - \gamma\phi(0)]\, e^{-\gamma t} + e^{-\gamma t} * \phi''$$

and so $\psi'(t) \geqslant 0$.

By Theorem 1.3.1, we have $0 < f(t) \leqslant \psi(t)$. From the original equation, we get that $f(t) \leqslant \phi(t)$.

EXAMPLES. (1) $K(t) = \ln(e^2 + t)$, $\phi(t) = t^2 + 2t + 1$, $\gamma = a/2 = 1$.

(2) $\phi(t) = (1 + t)^{3/2}$, $K(t)$ and γ as before.

(3) $K(t) = (4 + t)^{1/2}$, any of the above $\phi(t)$, γ as before.

The next theorem is on monotonicity of the solution. It would be assumed that (1) $K \in C^2$ for $0 \leqslant t < \infty$ and (2) $\phi \in C^3$ for $0 \leqslant t < \infty$.

THEOREM 1.3.4. *If* (1) $K(t) > 0$, $K'(t) > 0$, $K''(t) < 0$; (2) $a^2 \geqslant 4b$ *where* $a = K(0)$, $b = K'(0)$; (3) $\phi'(0) < \phi(0)\, a$, $\phi'''(t) \leqslant \phi(0)\, K''(t)$ *and* (4) *there exists a* γ *lying between the two positive numbers* $(a \pm (a^2 - 4b)^{1/2})/2$ *such that* $\phi''(0) - \gamma\phi'(0) \leqslant \phi(0)\,(b - \gamma a)$, *then* $f'(t) < 0$.

If the additional assumptions in Theorem 1.3.3 *also hold, then* $f(t)$ *is both positive and decreasing and so* $0 < f(t) \leqslant \phi(0)$.

Proof. The equivalent equation is $f(t) = \psi(t) - L * f$.

Again, if γ lies between the two zeroes of $\gamma^2 - a\gamma + b$, then $L(t) > 0$ and $L'(t) < 0$.

To apply Theorem 1.3.2, we have to show in addition that $\psi'(0) < \psi(0)\, L(0)$ and $\psi''(t) \leqslant \psi(0)\, L'(t)$.

We have that

$$\psi'(t) = [\phi'(0) - \gamma\phi(0)]\, e^{-\gamma t} + e^{-\gamma t} * \phi''$$

and $\psi(0) = \phi(0), L(0) = a - \gamma$, so we require

$$\phi'(0) - \gamma\phi(0) < \phi(0)\,(a - \gamma),$$

that is,

$$\phi'(0) < \phi(0)\, a$$

which is true by assumption.

We have

$$\psi''(t) = [-\gamma\phi'(0) + \gamma^2\phi(0) + \phi''(0)]\, e^{-\gamma t} + e^{-\gamma t} * \phi'''$$

and

$$\psi(0)\, L'(t) = \phi(0)\,(\gamma^2 - a\gamma + b)\, e^{-\gamma t} + \phi(0)\, e^{-\gamma t} * K'',$$

so $\psi''(t) \leqslant \psi(0)\, L'(t)$ since $\phi''(0) - \gamma\phi'(0) \leqslant \phi(0)\,(b - \gamma a)$ and $\phi'''(t) \leqslant \phi(0)\, K''(t)$.

EXAMPLE. $\phi(t) = (e^2 + t)\,[\ln(e^2 + t) - 1] - (e^2 - 1)$, $K(t) = \ln(e^2 + t) + 2t + 1$.

This example satisfies the assumptions in both Theorem 1.3.3 and Theorem 1.3.4.

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REFERENCES

1. R. BELLMAN AND K. L. COOKE, "Differential-Difference Equations," Academic, New York, 1963.
2. R. BELLMAN, R. KALABA, AND J. LOCKETT, "Numerical Inversion of the Laplace Transform," American Elsevier, New York, 1966.
3. A. FRIEDMAN, On integral equations of Volterra type, *J. Analyse Math.* 11 (1963), 381–413.
4. A. FRIEDMAN AND M. SHINBROT, Volterra integral equations in Banach space, *Trans. Amer. Math. Soc.* **126** (1967), 131–179.
5. F. G. TRICOMI, "Integral Equations," Interscience, New York, 1957.